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THE CRITICAL SEPARATION REYNOLDS NUMBER FOR STREAMING FLOW PAST A CIRCULAR CYLINDER

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Abstract—Employing a complex variable approach to the equations of motion for an incompressible viscous fluid and a more general approximation to the convecting stream function it is possible to calculate an approximation to the vorticity on the boundary for streaming flow past a circular cylinder without solving first for the complete flow field. In particular it is found that separation at the rear stagnation point first occurs when $R^* = 2.78$, where R^* is the critical Reynolds number. This result is in good agreement with the value of R^* obtained by experiment and the value obtained numerically. The convecting stream function satisfies the conditions of no slip and vorticity is not convected through the cylinder as it is in small Reynolds number Oseen theory.

1. INTRODUCTION

The problem of determining the steady streaming flow past a fixed circular cylinder is one of long standing interest in the theory of the two dimensional motion of an incompressible viscous liquid. It is well known that the Stokes approximation does not lead to a solution for the velocity field which converges to a uniform stream at infinity. The paradox of Stokes was rationalized by Oseen by taking into account the convection due to the uniform stream. The Oseen linearization of the equations of motion certainly leads to a solution for the velocity field which predicts correct behaviour at large distances but close to the cylinder vorticity is effectively transmitted through the cylinder by the uniform stream and although separation is predicted in a qualitative manner the value of the critical Reynolds number at which separation occurs is $R^* = 1.51$ and is too low compared with experimental results of Taneda (1956). The numerical value of R^* predicted by Underwood (1954) using a series truncation method is $R^* = 2.88$. The generalized series expansion of Skinner (1975) predict from three terms the value $R^* = 1$. This again may be considered too low and it is noted that the quantity $\Delta_i = [\log(4/R) - \gamma + \frac{1}{2}]^{-1}$ which occurs both in the Oseen solution and Skinner's work is positive for very small R but is negative at $R = 4$. Since Δ_1 must become infinite somewhere in the range $0 < R < 4$ it is not surprising the value of R^* is somewhat underestimated.

Dorrepaal (1982) has considered the Burgers linearization and has determined a value of R^* lying between the Oseen value and the value given by Skinner. It is noted that in Burger's flow the forced convection is produced by the potential flow past a cylinder and the resulting vorticity produced from the linearized equation is essentially convected around the cylinder since the no slip boundary condition is not satisfied by the forced convecting flow.

In this paper a theoretical analysis is presented for streaming flow past a fixed circular cylinder when the Reynolds number assumes its critical value for separation at the rear stagnation point. The two dimensional equations of motion are first set in complex form and an auxiliary function whose Laplacian is proportional to the total head of pressure is introduced so that the Navier Stokes equation can be written as a complex system. The equations are linearized by introducing a general convecting stream function and it is possible to integrate the system from fourth order to second order. Further integration is possible for an equation related to the pressure field. Even though the flow equations have been linearized it is a difficult problem to determine the flow field explicitly. However by applying inner and outer boundary conditions it is found that the vorticity on the boundary of the cylinder can be calculated. The convecting stream function is chosen to model the flow at low Reynolds number. Unlike the Oseen and Burgers flow this convecting stream function satisfies the inner and outer boundary conditions exactly and also contains some asymmetry so that separation can be predicted. With the knowledge of the boundary vorticity the major flow features can be discussed. First it is shown that separation occurs at a critical Reynolds number $R^* = 2.78$ in good agreement with the results of Underwood (1969) and Taneda (1956). In a slighly modified theory using the results from small Reynolds number hydrodynamics the value of R^* is found to be 2.72.

2. THE EQUATIONS OF MOTION AND METHOD OF SOLUTION

The Navier Stokes equations for steady incompressible flow are

$$(\mathbf{q}\cdot\nabla)\mathbf{q} = -\operatorname{grad}\frac{p}{\rho} + \nu\nabla^2\mathbf{q} \qquad [1]$$

$$\operatorname{div}\mathbf{q} = 0 \qquad [2]$$

where $\mathbf{q}$ is the fluid velocity, p the pressure, ρ the density and ν the kinematic viscosity. For two dimensional flow the velocity may be written as

$$\mathbf{q} = u(x, y)\hat{\imath} + v(x, y)\hat{\jmath} \qquad [3]$$

and introducing a stream function $\psi(x, y)$ by

$$u = -\psi_y, v = \psi_x \qquad [4]$$

the equation of continuity (2) is satisfied. Equation[1] may now be written as

$$-\psi_x\omega = -P_x - \nu\omega_y \qquad [5]$$

$$-\psi_y\omega = -P_y + \nu\omega_x \qquad [6]$$

where P is the total head of pressure defined by

$$P = \frac{p}{\rho} + \frac{1}{2}|\mathbf{q}|^2 \qquad [7]$$

and ω is the vorticity

$$\omega = \nabla_1{}^2\psi = \psi_{xx} + \psi_{yy}. \qquad [8]$$

Introducing complex variables z, $\bar{z}$ defined by

$$z = x + iy, \bar{z} = x - iy \qquad [9]$$

the complex fluid velocity can be expressed as

$$q = u + iv = 2i\psi_{\bar{z}} \qquad [10]$$

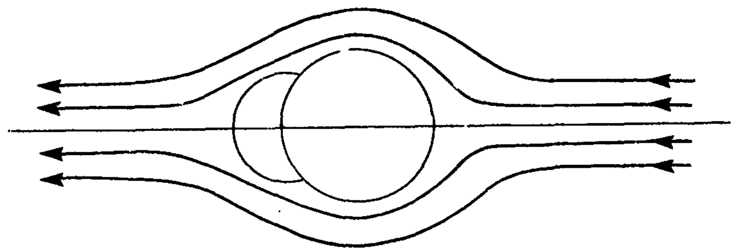

Figure 1. The streamlines for the convecting stream function β for $\alpha > \frac{1}{2}$.

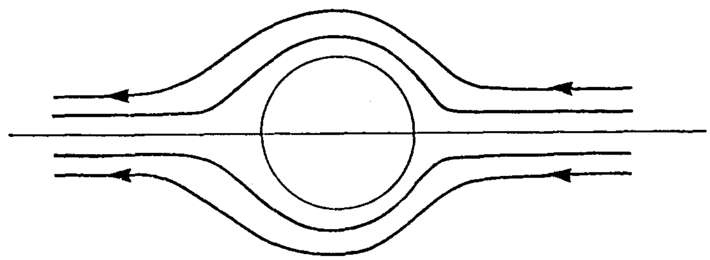

Figure 2. The streamlines for the convecting stream function β for $0 < \alpha < \frac{1}{2}$.

and the vorticity $\omega = 4\psi_{z\bar z}$ where

$$2\frac{\partial}{\partial \bar z} \equiv \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \quad 2\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}. \qquad [11]$$

If $\phi(x, y)$ is a real function defined by

$$P = -\nu\nabla_1{}^2\phi = -4\nu\phi_{z\bar z} \qquad [12]$$

and multiplying (6) by i and adding to [5], then [5] and [6] combine to yield

$$\gamma\phi_{\bar z\bar z z} + i\nu\psi_{\bar z\bar z z} + \psi_{\bar z}\psi_{z\bar z} = 0. \qquad [13]$$

Equation [13] admits one integration with respect to z and

$$\phi_{\bar z\bar z} + i\psi_{\bar z\bar z} + \frac{1}{2\nu}\psi_{\bar z}^2 = h''(\bar z) \qquad [14]$$

where the function $h(\bar z)$ is arbitrary. Without loss of generality ϕ may be replaced by $\phi + \bar h(z) + h(\bar z)$ and [14] may be replaced by the following equivalent form

$$\phi_{\bar z\bar z} + i\psi_{\bar z\bar z} + \frac{1}{2\nu}\psi_{\bar z}^2 = 0. \qquad [15]$$

Taking the complex conjugate of [15]

$$\phi_{zz} - i\psi_{zz} + \frac{1}{2\nu}\psi_z^2 = 0 \qquad [16]$$

and elimination of ϕ from [15] and [16] recovers the vorticity equation, viz.

$$\frac{\partial(\psi, \nabla_1{}^2\psi)}{\partial(x, y)} = \nu\nabla_1{}^4\psi. \qquad [17]$$

The Stokes flow equation obtained by the limit $\nu \to \infty$ from (15) is

$$\phi_{z\bar{z}} + i\psi_{z\bar{z}} = 0 \qquad [18]$$

and the general solution is

$$\phi + i\psi = \bar{z}f(z) + g(z). \qquad [19]$$

The Oseen linearization of the equations of motion can also be obtained from [15] and is of the form

$$\phi_{z\bar{z}} + i\psi_{z\bar{z}} - ik\psi_{\bar{z}} = 0 \qquad [20]$$

where k is a constant related to the free stream velocity and the kinematic viscosity. It is observed that [20] can be integrated once to give

$$\phi_{\bar{z}} + i\psi_{\bar{z}} - ik\psi = f(z). \qquad [21]$$

However, returning to the general equation [15] write

$$\phi' = \phi + \frac{1}{4\nu}\psi^2 \qquad [22]$$

then

$$\phi'_{z\bar{z}} = \phi_{z\bar{z}} + \frac{1}{2\nu}\psi\psi_{z\bar{z}} + \frac{1}{2\nu}\psi_z{}^2 \qquad [23]$$

and [15] may be written as

$$\phi'_{z\bar{z}} + \left(i - \frac{1}{2\nu}\psi\right)\psi_{z\bar{z}} = 0. \qquad [24]$$

Suppose $\psi = \beta$ is an approximation to the convecting stream function then [24] may be linearized to give

$$\phi'_{z\bar{z}} + \left(i - \frac{1}{2\nu}\beta\right)\psi_{z\bar{z}} = 0. \qquad [25]$$

Now consider the identity

$$\left(i - \frac{1}{2\nu}\beta\right)\psi_{z\bar{z}} = \frac{\partial}{\partial\bar{z}}\left\{\left(i - \frac{\beta}{2\nu}\right)\psi_z + \frac{1}{2\nu}\psi\beta_z\right\} - \frac{1}{2\nu}\psi\beta_{z\bar{z}} \qquad [26]$$

and [25] becomes

$$\phi'_{z\bar{z}} + \frac{\partial}{\partial\bar{z}}\left\{\left(i - \frac{\beta}{2\nu}\right)\psi_z + \frac{\psi\beta_z}{2\nu}\right\} - \frac{1}{2\nu}\psi\beta_{z\bar{z}} = 0. \qquad [27]$$

In the last term of [27] ψ is again replaced by β and

$$\frac{\partial}{\partial\bar{z}}\left\{\phi'_z + i\psi_z - \frac{1}{2\nu}\beta\psi_z + \frac{1}{2\nu}\psi\beta_z + \frac{1}{2\nu}\psi\beta_z\right\} = \frac{1}{2\nu}\beta\beta_{z\bar{z}}. \qquad [28]$$

Define $\chi = \phi' - 1/2\nu\beta\psi$, then

$$\frac{\partial}{\partial\bar{z}}\left\{\chi_{\bar{z}} + i\psi_{\bar{z}} + \frac{1}{\nu}\beta_{\bar{z}}\psi\right\} = \frac{1}{2\nu}\beta\beta_{\bar{z}\bar{z}}. \quad [29]$$

One integration with respect to $\bar{z}$ yields

$$\chi_{\bar{z}} + i\psi_{\bar{z}} + \frac{1}{\nu}\psi\beta_{\bar{z}} = \frac{1}{2\nu}\int\beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + f(z). \quad [30]$$

It is observed at this point that by making the one approximation ψ being replaced by β it is possible to integrate the equations of motion from fourth order to second order. To make further progress differentiate [30] partially with respect to z. This gives

$$\chi_{z\bar{z}} + i\psi_{z\bar{z}} + \frac{1}{\nu}\beta_{\bar{z}}\psi_z + \frac{1}{\nu}\psi\beta_{z\bar{z}} = \frac{1}{2\nu}\frac{\partial}{\partial z}\int\beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + f'(z). \quad [31]$$

Taking the real and imaginary parts of this equation

$$2\chi_{z\bar{z}} + \frac{1}{\nu}(\beta_{\bar{z}}\psi_z + \beta_z\psi_{\bar{z}}) + \frac{2}{\nu}\psi\beta_{z\bar{z}}$$
$$= \frac{1}{2\nu}\frac{\partial}{\partial z}\int\beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + \frac{1}{2\nu}\frac{\partial}{\partial\bar{z}}\int\beta\beta_{zz}\,\mathrm{d}z + f'(z) + \bar{f}'(\bar{z}) \quad [32]$$

and

$$2i\psi_{z\bar{z}} + \frac{1}{\nu}(\psi_z\beta_{\bar{z}} - \beta_z\psi_{\bar{z}}) = \frac{1}{2\nu}\frac{\partial}{\partial z}\int\beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} - \frac{1}{2\nu}\frac{\partial}{\partial\bar{z}}\int\beta\beta_{zz}\,\mathrm{d}z + f'(z) - \bar{f}'(\bar{z}). \quad (33]$$

It is not possible to approximate [33] satisfactorily without leading to the Stokes Paradox but in [32] ψ will be replaced by β throughout so that

$$2\chi_{z\bar{z}} + \frac{2}{\nu}(\beta_z\beta_{\bar{z}} + \beta\beta_{z\bar{z}}) = \frac{1}{2\nu}\frac{\partial}{\partial z}\int\beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + \frac{1}{2\nu}\frac{\partial}{\partial\bar{z}}\int\beta\beta_{zz}\,\mathrm{d}z + f'(z) + \bar{f}'(\bar{z}) \quad [34]$$

and integration yields

$$2\chi + \frac{\beta}{\nu} = \frac{1}{2\nu}\int\mathrm{d}\bar{z}\int\beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + \frac{1}{2\nu}\int\mathrm{d}z\int\beta\beta_{zz}\,\mathrm{d}z + \bar{z}f(z) + z\bar{f}(\bar{z}) + g(z) + \bar{g}(\bar{z}). \quad [35]$$

Differentiating partially with respect to $\bar{z}$

$$2\chi_{\bar{z}} + \frac{2}{\nu}\beta\beta_{\bar{z}} = \frac{1}{2\nu}\int\beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + \frac{1}{2\nu}\frac{\partial}{\bar{z}}\int\mathrm{d}z\int\beta\beta_{zz}\,\mathrm{d}z + f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}). \quad [36]$$

Consider now the specific problem of streaming flow past a fixed circular cylinder. If the cross-section of the cylinder is typified by $|z| = a$ and free stream speed is U it is appropriate to introduce nondimensional quantities by

$$z = az', \psi = Ua\psi', \chi = Ua\chi', \beta = Ua\beta', R = \frac{Ua}{\nu} \quad [37]$$

where the Reynolds number is based on the radius a.

Substitution of [37] and dropping primes it is found that the nondimensional forms of [30], [32], [33], [34], [36] are

$$\chi_{\bar{z}} + i\psi_{\bar{z}} + R\psi\beta_{\bar{z}} = \frac{R}{2}\int \beta\beta_{\bar{z}\bar{z}} + f(z). \qquad [38]$$

$$2\chi_{z\bar{z}} + R(\beta_{\bar{z}}\psi_z + \beta_z\psi_{\bar{z}}) + 2R\psi\beta_{z\bar{z}} = \frac{1}{2}R\frac{\partial}{\partial z}\int \beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + \frac{R}{2}\frac{\partial}{\partial \bar{z}}\int \beta\beta_{zz}\,\mathrm{d}z + f'(z) + \bar{f}'(\bar{z}). \qquad [39]$$

$$2i\psi_{z\bar{z}} + R(\psi_z\beta_{\bar{z}} - \beta_z\psi_{\bar{z}}) = \frac{R}{2}\frac{\partial}{\partial z}\int \beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} - \frac{1}{2}R\frac{\partial}{\partial \bar{z}}\int \beta\beta_{zz}\,\mathrm{d}z + f'(z) - \bar{f}'(\bar{z}). \qquad [40]$$

$$2\chi_{z\bar{z}} + 2R(\beta_z\beta_{\bar{z}} + \beta\beta_{z\bar{z}}) = \frac{1}{2}R\frac{\partial}{\partial z}\int \beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + \frac{1}{2}R\frac{\partial}{\partial \bar{z}}\int \beta\beta_{zz}\,\mathrm{d}z + f'(z) + \bar{f}'(\bar{z}). \qquad [41]$$

and

$$2\chi_{\bar{z}} + \frac{2}{1}R\beta\beta_{\bar{z}} = \frac{1}{2}R'\int \beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + \frac{1}{2}R\frac{\partial}{\partial \bar{z}}\int \mathrm{d}z\int \beta\beta_{zz}\,\mathrm{d}z + f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}). \qquad [42]$$

The inner boundary condition on the cylinder requires

$$\psi = \psi_{\bar{z}} = 0 \text{ at } |z| = 1 \qquad [43]$$

so that from [38]

$$\chi_{\bar{z}} = \frac{R}{2}\int \beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + f(z) \text{ on } |z| = 1. \qquad [44]$$

Elimination of $\chi_{\bar{z}}$ from (30) and (42) yields

$$\frac{R}{2}\frac{\partial}{\partial \bar{z}}\int \mathrm{d}z\int \beta\beta_{zz}\,\mathrm{d}z - \frac{R}{2}\int \beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) - f(z) = 0 \text{ on } |z| = 1. \qquad [45]$$

The outer boundary condition requires

$$\psi \sim r\sin\theta = \frac{1}{2}i(\bar{z} - z) \text{ as } |z| \to \infty \qquad [46]$$

or equivalently in terms of χ from [34]

$$\chi_{z\bar{z}} \sim \text{constants as } |z| \to \infty. \qquad [47]$$

Thus $f(z) \sim Bz - A/2\log z$ as $|z| \to \infty$ where A and B are real constants and the inner boundary condition [45] is satisfied by taking $\bar{g}'(\bar{z}) \sim +A/2\log\bar{z} + A/2\bar{z}^2$ as $|z| \to \infty$. Once β is known the complete functions f and g can be determined on the basis that the vorticity decays to zero at infinity. Desirable features of the convecting stream function β are that both inner and outer boundary conditions are satified, that is

$$\beta = \frac{\partial\beta}{\partial r} = 0 \text{ at } r = 1, z = r\,e^{i\theta} \text{ and}$$

$$\beta \sim r\sin\theta \text{ as } r \to \infty. \qquad [48]$$

Consider the following form for β

$$\beta = \frac{(r^2-1)^2 \sin\theta}{r^3} + \frac{\alpha(r^2-1)^2 \sin 2\theta}{r^4} \tag{49}$$

where α is a constant. The expression β satisfies the inner and outer boundary conditions and the vorticity on $r = 1$ derived from [49] is

$$\begin{aligned} \nabla_1{}^2\beta\,|_{r=1} &= 8\sin\theta + 8\alpha\sin 2\theta \\ &= 8\sin\theta(1 + 2\alpha\cos\theta). \end{aligned} \tag{50}$$

The flow described by β admits separation when $\alpha \geq \frac{1}{2}$, separation first occurring when $\theta = \pi$ and at $\alpha = \frac{1}{2}$. For the purpose of calculating the integrals it is convenient to write

$$\beta = \beta_0 + \beta_1 \tag{51}$$

where

$$\beta_0 = \frac{(r^2-1)^2 \sin\theta}{r^3}, \; \beta_1 = \frac{\alpha(r^2-1)^2 \sin 2\theta}{r^4}. \tag{52}$$

In terms of z and $\bar{z}$, β_0, β_1 may be expressed as

$$\beta_0 = \frac{1}{2i}\left\{z - \bar{z} - \frac{2}{\bar{z}} + \frac{2}{z} + \frac{1}{z\bar{z}^2} - \frac{1}{z^2\bar{z}}\right\} \tag{53}$$

and

$$\beta_1 = \frac{\alpha}{2i}\left\{\frac{z}{\bar{z}} - \frac{\bar{z}}{z} - \frac{2}{\bar{z}^2} + \frac{2}{z^2} + \frac{1}{z\bar{z}^3} - \frac{1}{\bar{z}z^3}\right\}. \tag{54}$$

Also the second derivatives are given by

$$\frac{\partial^2\beta_0}{\partial\bar{z}^2} = \frac{1}{2i}\left\{-\frac{4}{\bar{z}^3} + \frac{6}{z\bar{z}^4} - \frac{2}{z^2\bar{z}^3}\right\}. \tag{55}$$

$$\frac{\partial^2\beta_1}{\partial\bar{z}^2} = \frac{\alpha}{2i}\left\{\frac{2z}{\bar{z}^3} - \frac{12}{\bar{z}^4} + \frac{12}{z\bar{z}^5} - \frac{2}{\bar{z}^3z^3}\right\}. \tag{56}$$

Now the integral

$$\int \beta\beta_{\bar{z}\bar{z}}\,d\bar{z} = \int \beta_0 \frac{\partial^2\beta_0}{\partial\bar{z}^2}\,d\bar{z} + \int \beta_0 \frac{\partial^2\beta_1}{\partial\bar{z}^2}\,d\bar{z} + \int \beta_1 \frac{\partial^2\beta_0}{\partial\bar{z}^2}\,d\bar{z} + \int \beta_1 \frac{\partial^2\beta_1}{\partial\bar{z}^2}\,d\bar{z} \tag{57}$$

and

$$\begin{aligned} \int \beta_0 \frac{\partial^2\beta_0}{\partial\bar{z}^2}\,d\bar{z} = -\frac{1}{4}\Bigg\{ &-\frac{1}{\bar{z}}\left(4 + \frac{2}{z^2}\right) + \frac{1}{\bar{z}^2}\left(2z + \frac{8}{2} + \frac{2}{z^3}\right) \\ &- \frac{1}{3\bar{z}^3}\left(14 + \frac{20}{z^2} + \frac{2}{z^4}\right) + \frac{1}{\bar{z}}\left(\frac{4}{z} + \frac{2}{z^3}\right) - \frac{6}{5z^2\bar{z}^5}\frac{1}{}\Bigg\}. \end{aligned} \tag{58}$$

$$\int \beta_1 \frac{\partial^2 \beta_1}{\partial \bar{z}^2}\,d\bar{z} = -\frac{\alpha^2}{4}\left\{\frac{1}{\bar{z}}\left(2-\frac{2}{z^4}\right)-\frac{1}{\bar{z}^2}\left(\frac{8}{z}-\frac{2}{z^5}\right)\right.$$
$$\left.-\frac{1}{3\bar{z}^3}\left(2z^2-\frac{40}{z^2}+\frac{2}{z^6}\right)-\frac{1}{\bar{z}^4}\left(-4z+\frac{10}{z^3}\right)-\frac{1}{5\bar{z}^5}\left(38-\frac{14}{z^4}\right)+\frac{6}{z\bar{z}^6}-\frac{12}{7}\frac{1}{z^2\bar{z}^7}\right\}. \quad [59]$$

$$\int \beta_0 \frac{\partial^2 \beta_1}{\partial \bar{z}^2}\,d\bar{z} = -\frac{\alpha}{4}\left\{\frac{1}{\bar{z}}\left(\frac{2}{z^3}-2z\right)-\frac{1}{\bar{z}^2}\left(z^2+8-\frac{1}{z^2}-\frac{2}{z^4}\right)\right.$$
$$\left.-\frac{1}{3\bar{z}^3}\left(-16z-\frac{38}{z}+\frac{4}{z^3}+\frac{2}{z^5}\right)-\frac{1}{4\bar{z}^4}\left(38+\frac{36}{z^2}-\frac{2}{z^4}\right)+\frac{1}{5\bar{z}^5}\left(\frac{36}{z}+\frac{12}{z^3}\right)-\frac{2}{\bar{z}^6 z^2}\right\}. \quad [60]$$

$$\int \beta_1 \frac{\partial^2 \beta_0}{\partial \bar{z}^2} = -\frac{\alpha}{4}\left\{-\frac{1}{\bar{z}}\left(\frac{4}{z}+\frac{2}{z^3}\right)+\frac{1}{\bar{z}^2}\left(\frac{7}{z^2}+\frac{2}{z^4}\right)\right.$$
$$\left.-\frac{1}{3\bar{z}^3}\left(-4z+\frac{16}{z^3}-\frac{2}{z}+\frac{2}{z^5}\right)-\frac{1}{4\bar{z}^4}\left(14-\frac{6}{z^4}+\frac{4}{z^2}\right)+\frac{1}{5\bar{z}^5}\left(\frac{16}{z}+\frac{2}{z^3}\right)-\frac{1}{\bar{z}^6 z^2}\right\}. \quad [61]$$

It follows from [57] that

$$\int \beta\beta_{\bar{z}\bar{z}}\,d\bar{z} = -\frac{1}{4}\left\{\frac{1}{\bar{z}}\left(-4-\frac{2}{z^2}+2\alpha^2-\frac{2\alpha^2}{z^4}-\frac{4\alpha}{z^3}+2\alpha z-\frac{4\alpha}{z}\right)\right.$$
$$+\frac{1}{\bar{z}^2}\left(2z+\frac{8}{z}+\frac{2}{z^3}-\frac{8\alpha^2}{z}+\frac{2\alpha^2}{z^5}-\alpha z^2-8\alpha+\frac{8\alpha}{z^2}+\frac{4\alpha}{z^4}\right)$$
$$-\frac{1}{3\bar{z}^3}\left(14+\frac{20}{z^2}+\frac{2}{z^2}+\frac{2}{z^4}+2\alpha^2 z^2-\frac{40\alpha^2}{z^2}+\frac{2\alpha^2}{z^6}-\frac{20\alpha z}{1}-\frac{40\alpha}{z}+\frac{20\alpha}{z^3}+\frac{4\alpha}{z^5}\right)$$
$$-\frac{1}{\bar{z}^4}\left(\frac{4}{z}+\frac{2}{z^3}+4\alpha^2 z-\frac{10\alpha^2}{z^3}-13\alpha-\frac{10\alpha}{z^2}+\frac{2\alpha}{z^4}\right)$$
$$\left.+\frac{1}{5\bar{z}^5}\left(-\frac{6}{z^2}-38\alpha^2+\frac{14\alpha^2}{z^4}+\frac{52\alpha}{z}+\frac{14\alpha}{z^3}\right)+\frac{1}{\bar{z}^6}\left(\frac{6\alpha^2}{z}-\frac{3\alpha}{z^2}\right)-\frac{12\alpha^2}{7\bar{z}^7 z^2}\right\} \quad [62]$$

and the related integral

$$\frac{\partial}{\partial \bar{z}}\int dz \int \beta\beta_{zz}\,dz = -\frac{1}{4}\left\{\log z\left(\frac{4}{\bar{z}^3}+\frac{8\alpha^2}{\bar{z}^5}+\frac{12\alpha}{\bar{z}^4}+2\alpha+\frac{4\alpha}{\bar{z}^2}\right)\right.$$
$$-\frac{1}{z}\left(2-\frac{8}{\bar{z}^2}-\frac{6}{\bar{z}^2}-\frac{10\alpha^2}{\bar{z}^6}-2\alpha\bar{z}-\frac{16\alpha}{\bar{z}^3}-\frac{16\alpha}{\bar{z}^5}\right)$$
$$+\frac{1}{6\zeta^2}\left(-\frac{40}{\bar{z}^3}-\frac{8}{\bar{z}^5}+4\alpha^2\bar{z}^5+4\alpha^2\bar{z}+\frac{8\alpha^2}{\bar{z}^3}-\frac{12\alpha^2}{\bar{z}^7}-20\alpha+\frac{40\alpha}{\bar{z}^2}-\frac{60\alpha}{\bar{z}^4}\right.$$
$$\left.-\frac{20\alpha}{\bar{z}^6}\right)-\frac{1}{3z^3}\left(-\frac{4}{\bar{z}^2}-\frac{6}{\bar{z}^4}+4\alpha^2+\frac{30\alpha^2}{\bar{z}^4}+\frac{20\alpha}{\bar{z}^3}-\frac{8\alpha}{\bar{z}^5}\right)$$
$$\left.-\frac{1}{20z^4}\left(\frac{12}{\bar{z}^3}-\frac{56\alpha^2}{\bar{z}^5}-\frac{52\alpha}{\bar{z}^2}-\frac{42\alpha}{\bar{z}^4}\right)-\frac{1}{5z^5}\left(\frac{6\alpha}{\bar{z}^3}-\frac{6\alpha^2}{\bar{z}^2}\right)-\frac{4}{7}\frac{\alpha^2}{\bar{z}^3 z^6}\right\}. \quad [63]$$

From [62] and [63] it is found that

$$\left[\frac{\partial}{\partial\bar{z}}\int \mathrm{d}z\int \beta\beta_{zz}\,\mathrm{d}z-\int \beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z}\right]_{\substack{z=e^{i\theta}\\ \bar{z}=e^{i\theta}}}=-\frac{1}{4}\left\{e^{i\theta}(4-2\alpha^2)-\frac{3}{5}e^{-i\theta}+e^{2i\theta}\left(4\alpha i\theta+22\alpha-\frac{202\alpha}{15}\right)\right.$$
$$+\frac{7}{5}\alpha e^{-2i\theta}+e^{3i\theta}\left(4i\theta+\frac{10}{3}+\frac{6}{5}\right)$$
$$+\frac{22\alpha^2}{35}e^{-3i\theta}+e^{4i\theta}\left(12\alpha i\theta+13\alpha-\frac{2\alpha}{5}\right)$$
$$+e^{5i\theta}\left(8\alpha^2 i\theta-2\alpha^2+\frac{2\alpha^2}{3}+\frac{38\alpha^2}{5}+\frac{12\alpha^2}{7}\right)$$
$$\left.+2\alpha i\theta-6\alpha+\frac{21\alpha}{10}+\frac{20}{2}\alpha\right\}. \qquad [64]$$

Involing the inner boundary condition [45] on $|z|=1$ it is found that suitable forms for $f(z)$ and $\bar{g}'(z)$ are expressed by

$$\bar{g}'(\bar{z})=+\frac{A}{2}\log\bar{z}+\frac{A}{2\bar{z}^2}+\frac{a_1}{\bar{z}}+\frac{a_2}{\bar{z}^2}+\frac{a_3}{\bar{z}^3}+\frac{a_4}{\bar{z}^4}+\frac{a_5}{\bar{z}^5}+a$$
$$+\frac{c_2\log\bar{z}}{\bar{z}^2}+c_3\frac{\log\bar{z}}{\bar{z}^3}+c_4\frac{\log\bar{z}}{\bar{z}^4}+c^5\frac{\log\bar{z}}{\bar{z}^5} \qquad [65]$$

and

$$f(z)=Bz-\frac{A}{2}\log z+\frac{b_1}{z}+\frac{b_2}{z^2}+\frac{b_3}{z^3}+b_0\log z. \qquad [66]$$

The coefficients in [65] and [66] are found to be

$$a=\frac{R}{8}\left(-6\alpha+\frac{21\alpha}{10}+\frac{20\alpha}{3}\right) \qquad [67]$$

$$a_1=\frac{R}{8}(4-2\alpha^2),\ a_2=\frac{R}{8}\left(\frac{202\alpha}{15}-22\alpha\right)+\frac{\alpha R}{4} \qquad [68]$$

$$a_3=\frac{R}{8}\left(\frac{10}{3}+\frac{6}{5}\right)+\frac{3R}{40} \qquad [69]$$

$$a_4=\frac{R}{8}\left(13\alpha-\frac{2\alpha}{5}\right)-\frac{7\alpha R}{20} \qquad [70]$$

$$a_5=\frac{R}{8}\left(-2\alpha^2+\frac{2}{3}\alpha^2+\frac{38\alpha^2}{5}+\frac{12\alpha^2}{147}\right)-\frac{33\alpha^2 R}{140} \qquad [71]$$

$$b_0=-\frac{\alpha R}{4} \qquad c_2=-\frac{\alpha R}{2} \qquad [72]$$

$$b_1=\frac{3R}{40} \qquad c_3=-\frac{R}{2} \qquad [73]$$

$$b_2 = -\frac{7\alpha R}{40} \qquad c_4 = -\frac{3\alpha R}{2} \tag{74}$$

$$b_3 = -\frac{11\alpha^2 R}{140} \qquad c_5 = \frac{2\alpha^2 R}{2}. \tag{75}$$

It is noted that the constants A and B are as yet undetermined. A will be determined at a later stage but it is unnecessary to determine B for the vorticity on the boundary.

The vorticity on the cylinder $|z| = 1$ is from [40] expressed by

$$\omega i = 4i\psi_{z\bar{z}} = R\frac{\partial}{\partial z}\int \beta\beta_{\bar{z}\bar{z}}\,\mathrm{d}\bar{z} - R\frac{\partial}{\partial \bar{z}}\int \beta\beta_{zz}\,\mathrm{d}z + 2f'(z) - 2\bar{f}'(\bar{z}) \tag{76}$$

where the integrals are evaluated at $z = e^{i\theta}$. On calculation it is found that

$$\omega = \frac{13\alpha R}{15}\sin\theta + \frac{13R}{30}\sin 2\theta - \alpha R\sin 3\theta + \frac{R\alpha^2}{105}\sin 4\theta + 2A\sin\theta. \tag{77}$$

To determine the constant A it is noted that $\omega = 0 = \nabla_1{}^2\beta$ at $\theta = 0$ and $\theta = \pi$, and it is natural to identify ω and $\nabla_1{}^2\beta$ at a point midway between the forward and rear stagnation points, that is $\theta = \pm\pi/2$. It then follows from (50) and (77)

$$2A + \alpha R + \frac{13\alpha R}{15} = 8 \tag{78}$$

and ω can now be written as

$$= (8 - \alpha R)\sin\theta + \frac{13R}{30}\sin 2\theta - \alpha R\sin 3\theta + \frac{R\alpha^2}{105}\sin(4\theta). \tag{79}$$

Now the convecting stream function β first exhibits separation at the rear stagnation point of the cylinder when $\alpha = \frac{1}{2}$ and the critical Reynolds number R^* is obtained from the equation

$$\left.\frac{\partial\omega}{\partial\theta}\right|_{\theta=\pi} = 0 = -\left(8 - \frac{R^*}{2}\right) + \frac{13R^*}{15} + \frac{3}{2}R^* + \frac{R^*}{105} \tag{80}$$

giving R^* as

$$R^* = \frac{420}{151} = 2.78. \tag{81}$$

The photographic experiments of Taneda (1956) claim that twin vortices are not present at $R = 3$ but are observed at $R = 3.5$. Since there is difficulty in observing separation near the critical Reynolds, Taneda concluded that $R^* = 2.5$. The numerical valued based on series truncation given by Underwood (1969) is $R^* = 2.88$. However the theoretical value based on Oseen theory for small R is $R^* = 1.51$ (Yamada 1954) and the value derived from the three term inner expansion, again for small R, is $R^* = 1$ (Skinner 1975).

For values of $R > R^*$, the convecting vorticity $\nabla_1{}^2\beta|_{r=1}$ vanishes at $\theta = \pi$ and $\theta = \theta_0$, $\pi/2 < \theta_0 \leq \pi$ where

$$\cos\theta_0 = -\frac{1}{2\alpha}, \alpha \geq \frac{1}{2}. \tag{82}$$

Now if the vorticity ω also vanishes at $\theta = \theta_0$, then from [79] it follows that α can be expressed as a function of R in the form

$$\alpha = \frac{105}{2R}\left\{\left[16 + \frac{302R^2}{(105)^2}\right]^{1/2} - 4\right\}. \qquad [83]$$

The attachment angle θ_0 is now given as a function of R by

$$\cos\theta_0 = -\frac{1}{2\alpha} = -\frac{R}{105}\left\{\left[16 + \frac{302R^2}{(105)^2}\right]^{1/2} - 4\right\}^{-1}. \qquad [84]$$

Another possibility for choosing the coefficient A is to identify the term in $\sin\theta$ for the vorticity expression as the leading coefficient in the vorticity derived from the Stokes inner expansion for small Reynolds number. The reason for this is that the drag coefficient is derived entirely from the term in $\sin\theta$. Now the vorticity calculated from the leading term in the inner expansion is (see van Dyke 1975, p. 161)

$$\omega \sim 2\Delta_1,\ \Delta_1 = \frac{1}{\log 4 - \log R - \gamma + \frac{1}{2}} \qquad [85]$$

where γ is Euler's constant. Thus the coefficient A is given by

$$2A = \frac{2}{\log 4 - \log R - \gamma + \frac{1}{2}} - \frac{13\alpha R}{15} \qquad [86]$$

and the boundary vorticity is now expressed by

$$\omega = \frac{2}{[\log 4 - \log R - \gamma + \frac{1}{2}]}\sin\theta + \frac{13R}{30}\sin 2\theta - \alpha R\sin 3\theta + \frac{\alpha^2 R}{105}\sin 4\theta. \qquad [87]$$

The critical Reynolds number R^* at which separation first takes place is determined from the equation

$$\log 4 - \log R^* - \gamma + \tfrac{1}{2} = \frac{420}{499}\frac{1}{R^*}. \qquad [88]$$

The numerical value of R^* obtained from this equation is $R^* = 2.72$, which is in good agreement with the work of Taneda and Underwood. In the linearized theory presented in this paper vorticity is not convected through the boundary as it is in the Oseen approximation and this produces a more realistic value of R^*, more in line with the experimental and numerical values. Since the drag coefficient essentially depends on A it follows that for very small R the drag coefficient will be the same as in the Oseen approximation. Since only the approximate boundary vorticity is known it is not possible to calculate the eddy length for $R > R^*$. This requires a knowledge of the global velocity field which is beyond the scope of the present analysis. It is possible to calculate the vorticity derivative on the boundary $|z| = 1$ and this is expressed by

$$\begin{aligned} i\frac{\partial}{\partial r}(\nabla_1^2\psi)|_{r=1} &= R\frac{\partial}{\partial r}\left\{\frac{\partial}{\partial z}\int \beta\beta_{\bar z\bar z}\,d\bar z - \frac{\partial}{\partial \bar z}\int \beta\beta_{zz}\,dz\right\} + 2\frac{\partial}{\partial r}\{f'(z) - \bar f'(\bar z)\} \\ &= \frac{i17\alpha R}{15}\sin\theta - \frac{5Ri}{3}\sin 2\theta - \frac{\alpha Ri}{5}\sin 3\theta \\ &\quad + Ri\left(\frac{132}{15} - 4 - \frac{4}{21}\right)\alpha^2\sin 4\theta - \left(8 - \frac{28\alpha R}{15}\right)i\sin\theta. \end{aligned} \qquad [89]$$

Again the drag per unit length of cylinder can be calculated with the knowledge of the boundary vorticity and vorticity derivative. However, the convecting vorticity and its derivative do not agree with the corresponding values of the voriticity and vorticity derivative derived from [50] and [79] except at the forward and rear stagnation points. An accurate expression for the drag coefficient is therefore not possible without employing a more sophisticated form for β involving terms in $\sin 2\theta$, $\sin 3\theta$ and $\sin 4\theta$. Hence this is omitted from the present paper.

On the other hand it is noted that at the rear stagnation point $r = 1$, $\theta = \pi$

$$\psi = \beta = 0, \frac{\partial \psi}{\partial r} = \frac{\partial \beta}{\partial r} = 0 \tag{90}$$

$$\nabla_1^2 \beta = \nabla_1^2 \psi = 0, \frac{\partial}{\partial r} \nabla_1^2 \beta = \frac{\partial}{\partial r} \nabla_1^2 \psi = 0. \tag{91}$$

Also at $R = R^*$

$$\frac{\partial}{\partial \theta} \nabla_1^2 \beta = \frac{\partial}{\partial \theta} \nabla_1^2 \psi = 0. \tag{92}$$

Hence at $R = R^*$, ψ and β agree up to derivatives of third order at the rear stagnation and the value of R^* may be expected to be reasonably accurate.

Ideally the convecting vorticity on $r = 1$, should agree identically with the derived vorticity $\omega = \nabla_1^2 \psi$, on $r = 1$. This is not possible unless β is a solution of the Navier Stokes equations. To improve the approximation β may be expanded in the form

$$\beta = \frac{(r^2 - 1)^2}{r^3} \sin \theta + \sum_{m=2}^{n} \alpha_m \frac{(r^2 - 1)^2}{r^{m+2}} \sin m\theta. \tag{93}$$

This will lead to an expansion for the boundary vorticity ω in the form

$$\omega = \nabla_1^2 \psi = \sum_{m=1}^{2n} a_m \sin m\theta \tag{94}$$

where the coefficients a_m depend on α_m, $m = 1, n$ and R. Now the vorticity on $r = 1$, derived from [93] is

$$\nabla_1^2 \beta = 8 \sin \theta + 8 \sum_{m=2}^{n} \alpha_n \sin m\theta. \tag{95}$$

The coefficients α_m, $m = 1, \ldots, n$ are now determined by $a_m = 8\alpha_m$, $m = 2, \ldots, n$, $a_1 = 8$. However, the calculations of the integrals involving β and its derivatives are tedious and cumbersome and from a practical point of view it may not be possible to proceed higher than $n = 3$ or 4.

Finally, it has been assumed that the vorticity decays to zero at infinity. If [25] is differentiated twice with respect to z, it is found that

$$\phi'_{zz\bar{z}} + \left(i - \frac{1}{2\nu}\beta\right)\psi_{zz\bar{z}} - \frac{1}{\nu}\beta_z \psi_{z\bar{z}\bar{z}} - \frac{1}{2\nu}\beta_{z\bar{z}} = 0 \tag{96}$$

and taking the imaginary part

$$\mathrm{Im}\left(i\psi_{zz\bar{z}} - \frac{1}{\nu}\beta_z \psi_{z\bar{z}} - \frac{1}{2\nu}\beta_{zz}\psi_{\bar{z}}\right) = 0. \tag{97}$$

Since $\beta \sim \gamma = -\frac{1}{2}i(z-\bar{z})$ as $r \to \infty$, it follows that the vorticity at large distances satisfies the equation

$$\mathrm{Im}\left(i\omega_{z\bar{z}} + \frac{1}{2\nu} i\omega_{\bar{z}}\right) = 0$$

as $|z| \to \infty$, and $\omega = 4\psi_{z\bar{z}}$. This is an Oseen type of equation and it follows that the vorticity decays exponentially at infinity.

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